

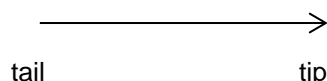
Physics 20 Lesson 10

Vector Addition

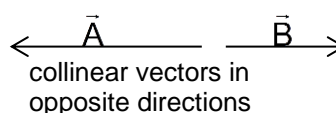
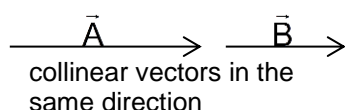
I. Vector Addition in One Dimension

(It is strongly recommended that you read pages 70 to 75 in Pearson for a good discussion on vector addition in one dimension.)

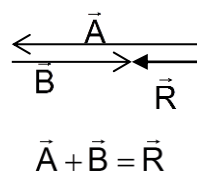
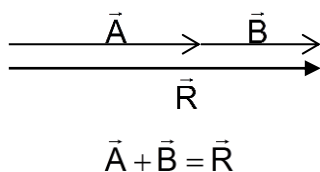
In previous lessons we used variables and graphs to represent vector quantities. Now we will learn to use vector diagrams to represent vector quantities. A vector is represented by a line segment with an arrowhead. Its beginning point is called the tail and the end point is called the tip. The length of a vector represents its magnitude and the arrow indicates the direction of the vector.



As we saw in Lesson 2, motion in one dimension involved vectors that are collinear (i.e. the vectors lie along the same line, either in the same or in opposite directions).



When adding vectors together one must add them tip-to-tail. For example, when two vectors are being added together the tail of the second vector is connected to the tip of the first vector. The sum of a series of vectors is called the resultant vector (\vec{R}). The resultant vector is drawn from the tail of the first vector to the tip of the last vector. For the collinear vectors above the vector additions are:



Note: It is important to understand that only the same type of vectors with the same units can be added together. One cannot, for example, add a 10 km east vector to a 15 m/s east vector.

You can add vectors graphically and algebraically. In Example 1 we will add three collinear vectors – first graphically and then algebraically.

Example 1

A person walks 20 m west, then 15 m east, and finally 10 m east. What is the person's final displacement?

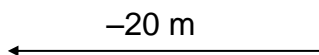
Graphical vector addition

1. Choose an appropriate scale and choose a reference direction.

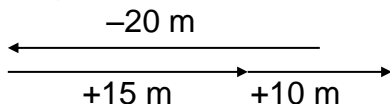
scale 1.0 cm : 5.0 m

direction east is positive, west is negative

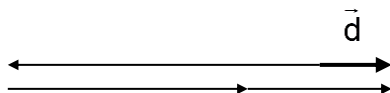
2. Draw the first vector and label its magnitude and direction



3. Place the tail of the second vector at the tip of the first vector. Continue to place all the remaining vectors in order tip-to-tail.



4. The resultant vector (\vec{d}) is drawn from the tail of the first vector to the tip of the last vector.



Find the magnitude of the resultant vector by measuring with a ruler, then convert the measured value using the scale. Remember to include the direction.

$$\vec{d} = 1.0 \text{ cm east} \times \frac{5.0 \text{ m}}{1.0 \text{ cm}}$$

$$\vec{d} = 5.0 \text{ m east}$$

Algebraic vector addition

1. Choose a reference direction.

direction east is positive, west is negative

2. Add the vectors together using the appropriate directions. (Note that this simple addition of positive and negative numbers works for collinear vectors only.)

$$\vec{R} = \vec{d} = \vec{A} + \vec{B} + \vec{C}$$

$$\vec{d} = 20 \text{ m west} + 15 \text{ m east} + 10 \text{ m east}$$

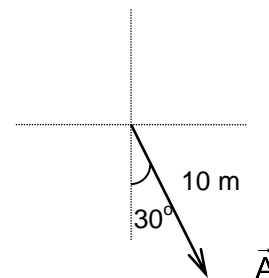
$$\vec{d} = 5.0 \text{ m east}$$

II. Vector Addition in Two Dimensions

(It is strongly recommended that you read pages 76 to 82 in Pearson for a good discussion on vector addition in two dimensions.)

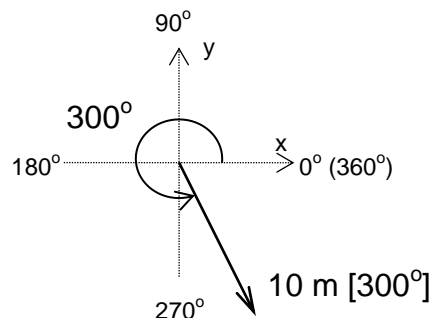
Vector addition in two dimensions is very similar to adding vectors in one dimension in that we use the tip-to-tail method. However, vector addition in two dimensions results in the formation of triangles which require the use of the Pythagorean formula and trigonometric functions to solve them. (For a review of the Pythagorean formula and trigonometric functions see the **Review of Trigonometric Functions** section below.)

In addition, when a vector has a direction which is not directly north, south, east or west we require a way to communicate the **direction** of the vector. Consider vector \vec{A} shown to the right. We can communicate its direction in several ways and we will use two methods:



Cartesian method

With the Cartesian method the positive x-axis is at 0° and angles are measured by rotating counter clockwise about the origin. For vector \vec{A} the angle is 300° from the positive x-axis.



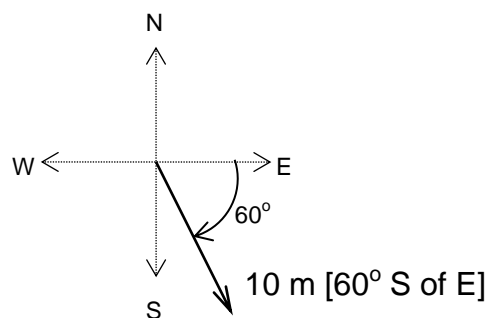
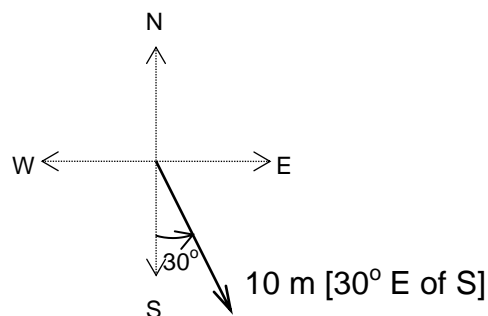
Navigator method

The navigator method uses the compass bearings north [N], south [S], east [E] and west [W]. The angle is stated as an angle rotating away from a starting direction. In this example, the 30° angle starts from the south and rotates toward the east. The angle is stated as "30 degrees east of south".

Note that the direction that the angle starts from is stated last.

Also notice that we can state the direction for the vector as an angle starting from the east direction. The angle would be "60 degrees south of east".

In other words, 10 m $[30^\circ \text{ E of S}]$ is equivalent to 10 m $[60^\circ \text{ S of E}]$.



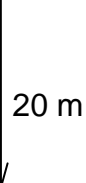
To illustrate vector addition in two dimensions, consider the following examples.

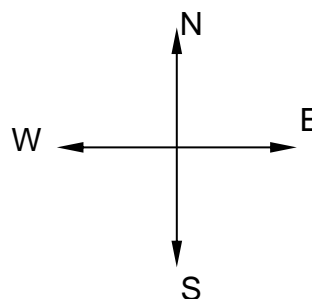
Example 2

A person walks 10 m east and then 20 m south. What is her final displacement?

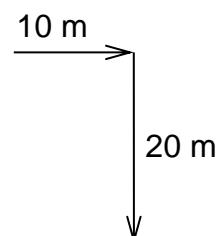
1. The vectors should be drawn to indicate their relative lengths – the 20 m vector is drawn twice as long as the 10 m vector. In addition, each vector should be drawn with an arrow to indicate its direction.

$\vec{A} = 10 \text{ m east}$ 

$\vec{B} = 20 \text{ m south}$ 

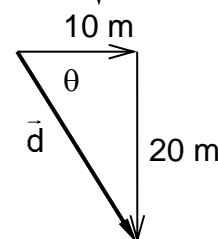


2. When adding the vectors together ($\vec{A} + \vec{B}$) the *tail* of the second vector is drawn from the *tip* of the first vector.



3. The resultant vector (\vec{d}) is drawn from the starting point to the tip of the final vector.

$$\vec{A} + \vec{B} = \vec{d}$$



4. To fully describe the resultant vector we need to calculate its magnitude and direction. Using the Pythagorean equation we can calculate the magnitude of \vec{d} .

$$\begin{aligned} c^2 &= a^2 + b^2 \\ d^2 &= 10^2 + 20^2 \\ d &= \sqrt{10^2 + 20^2} \\ d &= 22.4 \text{ m} \end{aligned}$$

5. The direction of (\vec{d}) is described by calculating the angle (θ) of the resultant vector. Since we have the opposite and adjacent sides of the triangle we use the tangent function to calculate the angle.

$$\begin{aligned} \tan \theta &= \frac{\text{opp}}{\text{adj}} \\ \theta &= \tan^{-1} \frac{\text{opp}}{\text{adj}} \end{aligned}$$

6. State the final displacement.

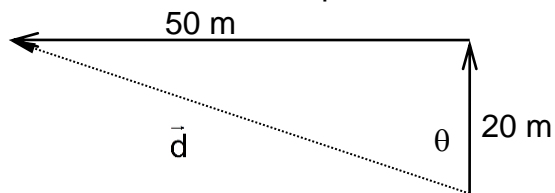
$$\vec{d} = 22.4 \text{ m} [63^\circ \text{ S of E}] \text{ or } 22.4 \text{ m } [297^\circ]$$

$$\begin{aligned} \theta &= \tan^{-1} \frac{20}{10} \\ \theta &= 63^\circ \end{aligned}$$

Example 3

A man walks 20 m north and then 50 m west.

- A. What is the distance traveled?
B. What is the man's final displacement?



- A. distance

$$\Delta d = 20 \text{ m} + 50 \text{ m} = \mathbf{70 \text{ m}}$$

- B. displacement (magnitude and direction from starting point)

$$d = \sqrt{20^2 + 50^2} = 53.8 \text{ m}$$

$$\theta = \tan^{-1} \frac{\text{opp}}{\text{adj}} = \tan^{-1} \frac{50}{20} = 68.2^\circ \text{ W of N}$$

$$\vec{d} = \mathbf{53.8 \text{ m } [68.2^\circ \text{ W of N}]}$$

or

$$\vec{d} = \mathbf{53.8 \text{ m } [158.2^\circ]}$$

Example 4

An object travels 50 m west, 30 m north, 150 m east and 10 m south. What is the resulting displacement?

Rather than drawing out four vectors tip-to-tail, we can reduce the problem by first finding the net displacement in each dimension:

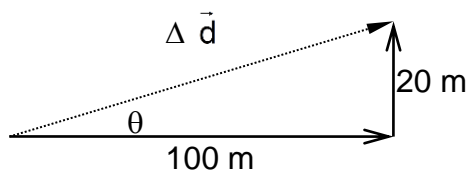
(North - South)

$$30 \text{ m North} + 10 \text{ m South} = 20 \text{ m North}$$

(East - West)

$$50 \text{ m West} + 150 \text{ m East} = 100 \text{ m East}$$

Draw a vector diagram of the displacements in each dimension.



$$d = \sqrt{20^2 + 100^2} = 102 \text{ m}$$

$$\theta = \tan^{-1} \frac{\text{opp}}{\text{adj}} = \tan^{-1} \frac{20}{100} = 11.3^\circ \text{ N of E}$$

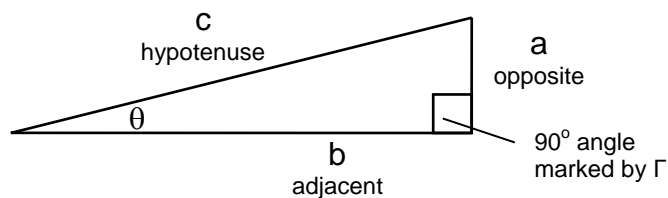
$$\vec{d} = 102 \text{ m } [11.3^\circ \text{ N of E}] \text{ or } \vec{d} = 102 \text{ m } [78.7^\circ \text{ E of N}] \text{ or } \vec{d} = 102 \text{ m } [11.3^\circ]$$

III. Practice Problems

1. An ant crawls 40 cm south and then 20 cm west. What is the final distance and displacement for the ant? (60 cm, 44.7 cm @ 63° south of west)
2. A car is driven 80 km north, then 50 km east, then 30 km south and then 90 km west. The entire trip required 4.5 hours. What was the average speed and displacement of the car? (56 km/h, 64 km @ 51° N of W)

IV. Review of Trigonometric Functions

In previous mathematics courses you have considered and studied the geometry of *right angle triangles*. Consider the diagram of a right angle triangle where the right angle (90°) is indicated and the angle of interest is (θ).



Recall the pythagorean theorem

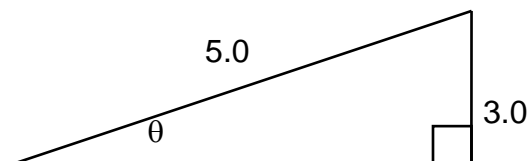
$$c^2 = a^2 + b^2$$

and the trigonometric ratios (soh cah toa).

$$\left. \begin{array}{l} \sin \theta = \frac{\text{opp}}{\text{hyp}} \\ \cos \theta = \frac{\text{adj}}{\text{hyp}} \\ \tan \theta = \frac{\text{opp}}{\text{adj}} \end{array} \right\} \text{and to calculate } \theta \text{ from the ratio} \left\{ \begin{array}{l} \theta = \sin^{-1} \left(\frac{\text{opp}}{\text{hyp}} \right) \\ \theta = \cos^{-1} \left(\frac{\text{adj}}{\text{hyp}} \right) \\ \theta = \tan^{-1} \left(\frac{\text{opp}}{\text{adj}} \right) \end{array} \right.$$

Example 5

For the following triangle, calculate the angle θ .



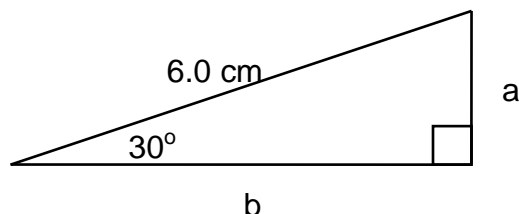
We are given the values for the hypotenuse (5.0) and the opposite side (3.0). Therefore we use the sine function.

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\theta = \sin^{-1} \left(\frac{\text{opp}}{\text{hyp}} \right) = \sin^{-1} \left(\frac{3.0}{5.0} \right) = 36.9^\circ$$

Example 6

Consider the diagram of a right triangle below, determine the other sides of the triangle.



To find side a we can use the sine function. (We are given the hypotenuse and we are asked to find the opposite side.)

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \text{by re-arrangement we get } \text{opp} = \text{hyp} \times \sin \theta$$

$$a = 6.0 \text{ cm} \times \sin 30^\circ = 3.0 \text{ cm}$$

To find side b we can use the cosine function. (We are given the hypotenuse and we are asked to find the adjacent side.)

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \text{by re-arrangement we get } \text{adj} = \text{hyp} \times \cos \theta$$

$$b = 6.0 \text{ cm} \times \cos 30^\circ = 5.2 \text{ cm}$$

V. Trigonometry Review

The questions below are optional and they should be done by those who wish to sharpen and review their skills in using trigonometry.

1. Draw a right angle triangle and label all sides (a , b , c) and angles (A , B) ($C = 90^\circ$). For angle A write out the trigonometric functions: sine, cosine, and tangent. According to Pythagoras, what is the length of the hypotenuse (side c)?
2. Find the number of degrees in each of the following angles: (a) $\sin \theta = 0.456$ (b) $\sin \theta = 0.748$ (c) $\cos \theta = 0.677$ (d) $\cos \theta = 0.224$ (e) $\tan \theta = 12.736$
3. Find the values of the following: $\sin 30$, $\sin 45$, $\sin 60$, $\cos 30$, $\cos 45$, $\cos 60$, $\tan 30$, $\tan 45$, $\tan 60$
4. Given a right triangle with an hypotenuse length of 42 cm and an angle of 30° . What are the lengths of the opposite and adjacent sides? (21 cm, 36.4 cm)
5. One angle of a right triangle is 55° and the adjacent side with length 24 m, determine the length of the hypotenuse. (41.8 m)
6. In a right triangle there is an angle of 38° opposite a side with a length of 15 m. What is the length of the hypotenuse? (24.4 m)
7. The opposite and adjacent sides of a right triangle are 12 cm and 16 cm respectively. What is the angle involved? What is the length of the hypotenuse? (36.9° , 20 m)
8. The hypotenuse of an isosceles right triangle is 20 cm long. What is the length of the other two sides? What are the other two angles? (14.1 m, 14.1 m, 45° , 45°)

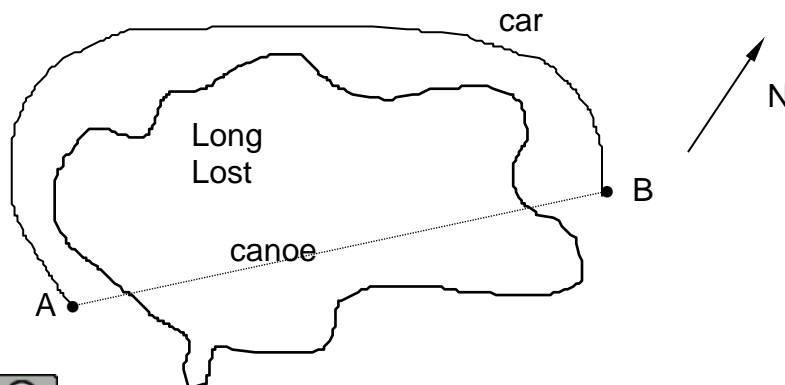
VI. Hand-in Assignment

For questions 1 to 4 do each of the following:

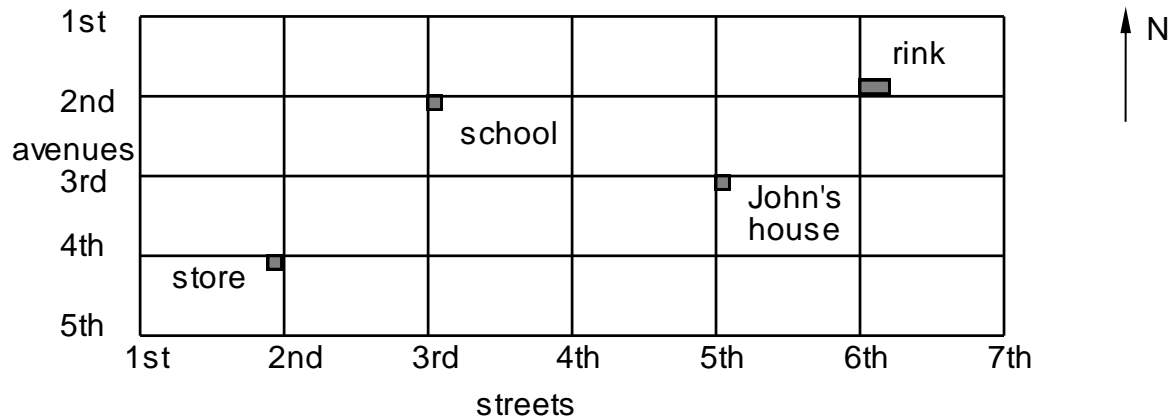
(i) Draw a scale diagram (ii) Find distance (iii) Find displacement

1. An student walks 20 m [N] then 10 m [S] then 30 m [N] then 25 m [S]. (85 m, 15 m [N])
2. An ant walks 20 cm [W] then 5 cm [N]. (25 cm, 20.6 cm [14° N of W])
3. A mouse runs 30 m [W], 10 m [N], and then 20 m [W]. (60 m, 51 m [11° N of W])
4. A hamster runs 60 cm [N], 30 cm [E], and then 90 cm [S]. (180 cm, 42.4 cm [45° E of S])

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5. What is the displacement after a plane flies: 20 km [W], 50 km [S] and 60 km [E]? (64.0 km [51.3° S of E])
 6. An object travels 858 m WEST then 297 m SOUTH. The time required for the journey was 13 minutes.
 - A. What was the total distance traveled? (1155 m)
 - B. What was the displacement of the object? (908 m [19° S of W])
 - C. What was the average speed? (88.8 m/min)
 7. An object travels 500 m south, 480 m east, 350 m north, 1650 m west, 390 m north, 590 m east, and 190 m north. The entire trip required 3 hours and 21 minutes.
 - A. What was the resulting displacement of the object? (722 m [53° W of N])
 - B. What was the average speed (in m/minute)? (20.6 m/min)
 8. The sketch below represents two journeys - one by car, another by canoe. According to the odometer, the car went 48 km to travel from A to B. The time required was 1.2 hours. Prospector Ralph went from A to B by canoe in exactly the same time. He found that during the trip he was always paddling along a line at 20° East of North. Ralph paddled a distance of 12 km.
 - A. Give the distance traveled in km by: i) the car, and ii) the canoe.
 - B. Give the final displacement in km for: i) the car, and ii) the canoe.
 - C. What was the average speed in km/h for: i) the car, and ii) the canoe.



9. Use the diagram below to answer parts A to C. (Streets are 0.75 km apart and avenues are 0.50 km apart.)



- If Bill walks from school to John's house, what is the displacement from the school to John's house? (1.6 km [18° S of E])
- If Jane walks from the store to John's house, what is her displacement from the store to John's house? (2.3 km [12.5° N of E])
- If Hank walks from the rink to the store, what was his displacement? (3.2 km [72° W of S])