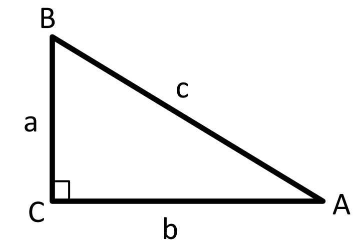
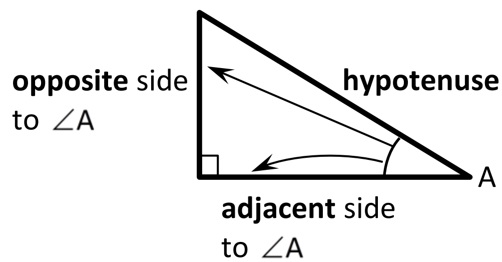
Math 10

<u>Lesson 7–1</u> Trigonometry – tangent, sine, cosine

I. Trigonometry – An overview of the basics



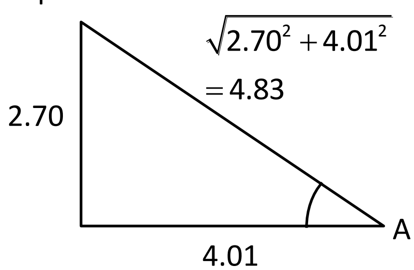
Way back in the days of the ancient Greeks and beyond, people like Euclid, Pythagoras, Archimedes, and many others studied triangles and the relationship between their angles and sides. Recall that for a right angle triangle we label the sides with a, b and c and we label the angles with A, B and C. Note that ∠A is the angle **opposite** from side a and that ∠B is the angle opposite from side b.

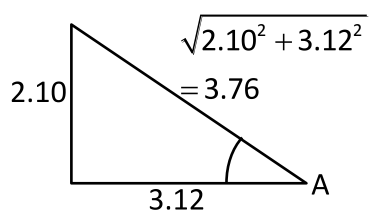


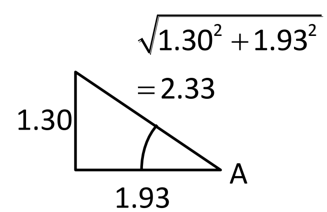
Another way to represent the triangle is shown to the right. The long side of the triangle is called the **hypotenuse** while the legs of the triangle are referred to as the **opposite** and **adjacent** sides to an angle. Opposite means "across from" and adjacent means "beside". Hence, the leg across from ∠A is called the "opposite side to ∠A" and the leg beside ∠A is called the "adjacent side to ∠A."

In previous courses you learned about the Pythagorian equation ($c^2 = a^2 + b^2$) and how to use it. But the Greeks also noticed another interesting property of right triangles. They noticed that for the angles of a right triangle the **ratios** of $\frac{\text{opposite}}{\text{adjacent}}$ and $\frac{\text{opposite}}{\text{hypotenuse}}$ and $\frac{\text{adjacent}}{\text{hypotenuse}}$ were **always the same** for the same angle.

To illustrate, consider the following three triangles that have the same angle A that, in this case, equals 34^o. The ratios are shown below.







$\frac{\text{opposite}}{\text{adjacent}}=\frac{2.70}{4.01}=\mathbf{0.67}$ $\frac{\text{opposite}}{\text{adjacent}}=\frac{2.10}{3.12}=\mathbf{0.67}$ $\frac{\text{opposite}}{\text{adjacent}}=\frac{1.30}{1.93}=\mathbf{0.67}$

$\frac{\text{opposite}}{\text{hypotenuse}}=\frac{2.70}{4.83}=\mathbf{0.56}$ $\frac{\text{opposite}}{\text{hypotenuse}}=\frac{2.10}{3.76}=\mathbf{0.56}$ $\frac{\text{opposite}}{\text{hypotenuse}}=\frac{1.30}{2.33}=\mathbf{0.56}$

$\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{4.01}{4.83}=\mathbf{0.83}$ $\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{3.12}{3.76}=\mathbf{0.83}$ $\frac{\text{adjacent}}{\text{hypotenuse}}=\frac{1.93}{2.33}=\mathbf{0.83}$

The important thing to grasp is that *the same ratio value emerges for the same angle no matter how large or how small the triangle*.

The Greeks gave special names for each of the ratios:

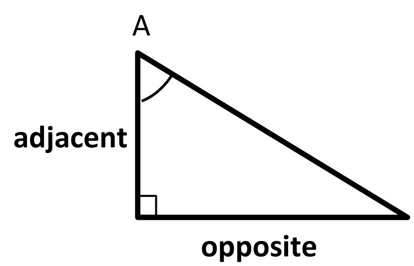
sine = $\dfrac{\text{opposite}}{\text{hypotenuse}}$ **cosine** = $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ **tangent** = $\dfrac{\text{opposite}}{\text{adjacent}}$

In other words, sine = opposite over hypotenuse (SOH)
cosine = adjacent over hypotenuse (CAH)
tangent = opposite over adjacent (TOA)

remember: **SOH CAH TOA**

II. The tangent ratio

In this section of the lesson we will talk about the tangent function. In the next section we will revisit the sine and cosine ratios. As we saw above, the tangent ratio is the ratio of the opposite over the adjacent sides of the triangle and it depends only on the measure of the angle, not on how large or small the triangle is.

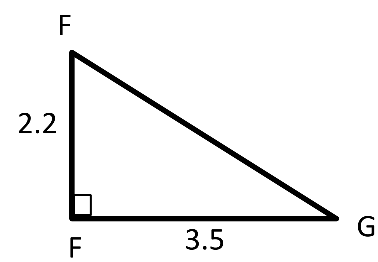


The tangent ratio for ∠A is written as **tan A**. We usually write the tangent ratio as one of the following, depending on which is more convenient at the time.

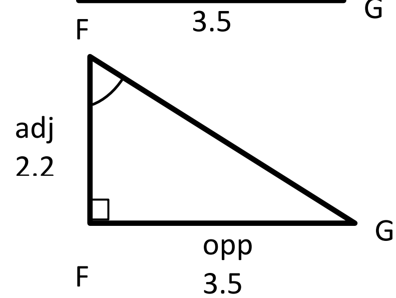
$$\tan A = \frac{\text{opposite}}{\text{adjacent}} \qquad \tan A = \frac{\text{opp}}{\text{adj}} \qquad \tan A = \frac{o}{a}$$

Calculating the ratio

The value of the tangent ratio is usually expressed as a decimal that compares the lengths of the sides. For example, consider the following triangle with ∠F and ∠G.



To find the tangent of ∠F we first identify which side is opposite and which side is adjacent to ∠F. Then we calculate the tangent

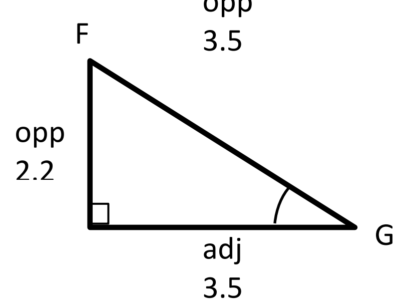


$$\tan F = \frac{\text{opp}}{\text{adj}}$$

$$\tan F = \frac{3.5}{2.2}$$

$$\tan F = \mathbf{1.591}$$

To find the tangent of ∠G we identify the opposite and adjacent sides and then calculate



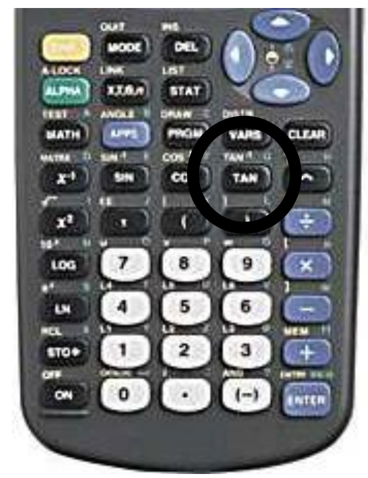
$$\tan G = \frac{\text{opp}}{\text{adj}} = \frac{2.2}{3.5} = \mathbf{0.6286}$$

Calculating the angle

Recall that the ratio between sides depends on the angle and not on the size of the triangle. Therefore, every ratio that we calculate corresponds to a particular angle. In the days of the ancient Greeks, long before calculators were invented, mathematicians would use a ruler and a protractor and they would draw triangles with different angles. For each degree from 0 to 90 they would measure the sides of the triangle and then calculate the tangent ratio. The information was put into a table like the one on the right so that people could look up the ratio and the angle that corresponded to that ratio. Fortunately for us we can use a calculator to find (a) the angles that correspond to a ratio and (b) to calculate the tangent from the angle.

tangent ratio	angle
0	0^o
0.1763	10^o
0.3640	20^o
0.5774	30^o
0.8391	40^o
1.1918	50^o
1.7321	60^o
2.7475	70^o
5.6713	80^o
∞	90^o

From the example above we calculated **tan F = 1.591** and **tan G = 0.6286**. To find the angles that correspond to these ratios we calculate the **inverse tangent**. The symbol for inverse tangent on your calculator is **$\tan^{-1}$** and it is found as letters above the TAN button. To find ∠F, for example, we key in 2nd, TAN, 1.591, ENTER. **TRY IT NOW!!** The screen should look something like this:



$\tan^{-1}(1.591)$ 57.84917994	$\tan^{-1}(1.591)$ 1.009658659 If you are getting this result your calculator is in the wrong mode. Press MODE and change **radians** to **degrees**.

Now calculate ∠G. Your calculator should now look like:

$\tan^{-1}(1.591)$
57.84917994
$\tan^{-1}(.6286)$
32.15346853

on paper we would write

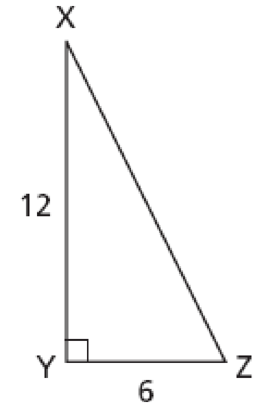
$\tan F = 1.591$ | $\tan G = .6286$
$F = \tan^{-1}(1.591)$ | $G = \tan^{-1}(.6286)$
$F = 57.8^o$ | $G = 32.2^o$

Question 1

For the given triangle:

a) Determine tan X and tan Z.

b) Calculate ∠X and ∠Z to the nearest tenth of a degree.



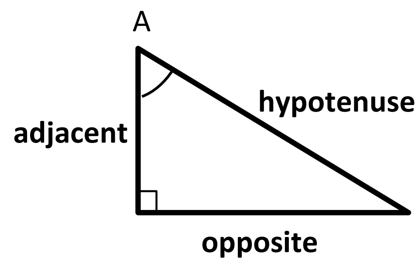
Question 2

Calculate the tangent for 25^{o} and for 73^{o}.

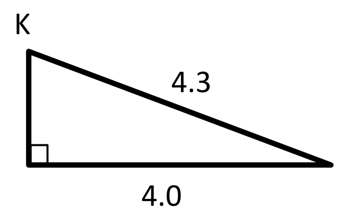
III. The sine and cosine ratios

The sine and cosine ratios are very similar to the tangent ratio. As we saw above, the sine of an angle is the ratio of the opposite over the hypotenuse, while the cosine of an angle is the ratio of the adjacent over the hypotenuse.

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \cos A = \frac{\text{adj}}{\text{hyp}}$$



To illustrate the use of the sine and cosine ratios, consider ∠K in the triangle to the right. Let's calculate K to one decimal place. We look to see what the sides are in relation to K. 4.3 is the hypotenuse and 4.0 is opposite (i.e. across from) K. The function that involves opposite and hypotenuse is the sine function. Therefore:



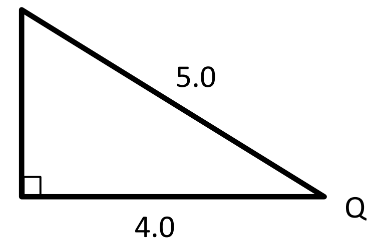
$$\sin K = \frac{\text{opp}}{\text{hyp}} = \frac{4.0}{4.3} = 0.9302$$

$$K = \sin^{-1} 0.9302$$

$$K = \mathbf{68.5^{o}}$$

Question 3

For the given triangle calculate ∠Q to the nearest tenth of a degree.



Question 4

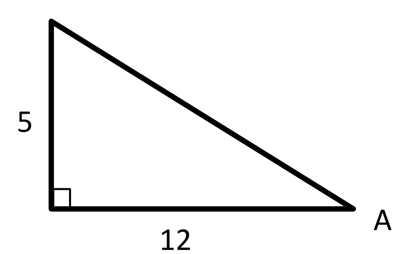
Calculate sin 30°, cos 30° and tan 30°.

Example 1 Angle of inclination

Dr. Licht has a cabin near Nelson, British Columbia that he built himself. The roof has a pitch of 5:12 which means that the roof rises 5’ for every 12’ of run. Determine the angle of inclination of the roof to the nearest tenth of a degree.

Solution

First we draw a diagram of the situation described.
We are given the opposite side and the adjacent side.
Since opp and adj involve the tan function we have:



$$\tan A = \frac{\text{opp}}{\text{hyp}} = \frac{5}{12} = 0.41\overline{6}$$

$$A = \tan^{-1} 0.41\overline{6}$$

$$A = \mathbf{22.6^\circ}$$

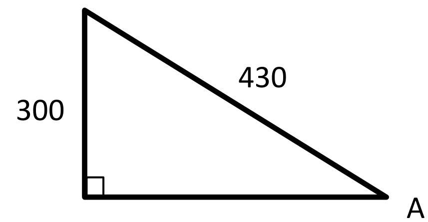
The angle of inclination is 22.6°.

Example 2 Angle of elevation

An observer is sitting on a dock watching a float plane in Vancouver harbour. At a certain time the plane is 300 m above the water and 430 m from the observer. Determine the angle of elevation of the plane measured from the observer to the nearest degree.

Solution

First we draw a diagram of the situation described. We are given the opposite side and the hypotenuse of the angle of elevation. Since opp and hyp involve the sin function we have:



$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{300}{430} = 0.69767$$

$$A = \sin^{-1} 0.69767$$

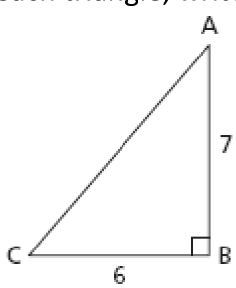
$$A = \mathbf{44^{o}}$$

The angle of elevation is 44^{o}.

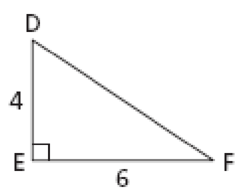
IV. Assignment

1. In each triangle, write the tangent ratio for each acute angle.

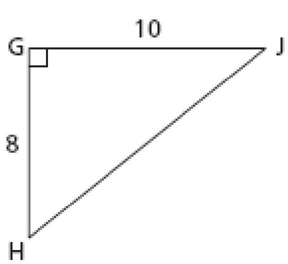
a)



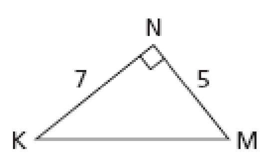
b)



c)



d)



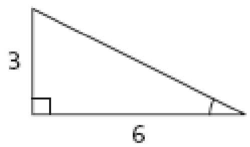
2. To the nearest degree, determine the measure of ∠X for each value of tan X.

a) tan X = 0.25 b) tan X = 1.25

c) tan X = 2.50 d) tan X = 20

3. Determine the measure of each indicated angle to the nearest degree.

a)



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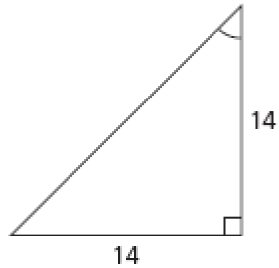

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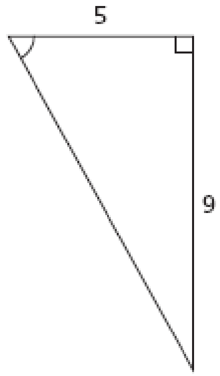
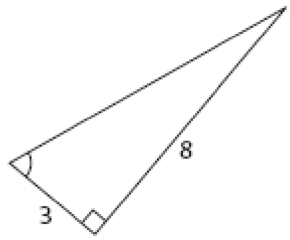


d)



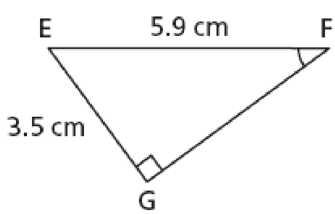
4. Use grid paper. Illustrate each tangent ratio by sketching a right triangle, then labelling the measures of its legs.

a) $\tan B = \frac{3}{5}$ b) $\tan E = \frac{5}{3}$

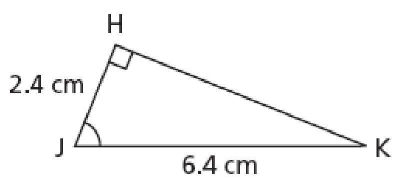
c) $\tan F = \frac{1}{4}$ d) $\tan G = 4$

5. Determine the measure of each indicated angle to the nearest tenth of a degree.

a)

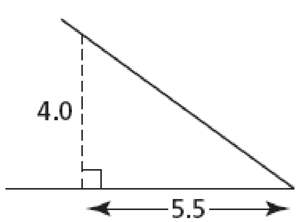


b)



6. Determine the angle of inclination of each line to the nearest tenth of a degree.

a)



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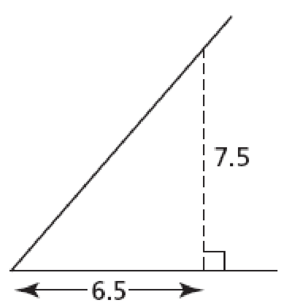
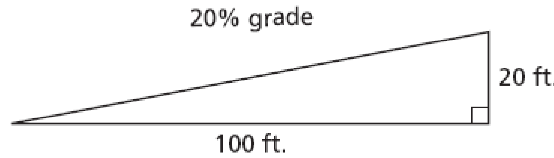


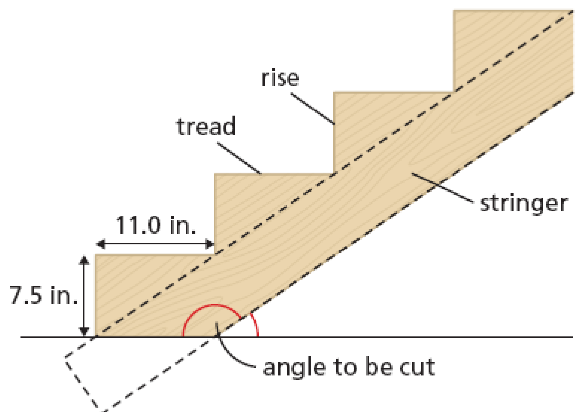
7. The grade or inclination of a road is often expressed as a percent. When a road has a grade of 20%, it increases 20 ft. in altitude for every 100 ft. of horizontal distance. Calculate the angle of inclination, to the nearest degree, of a road with each grade. a) 20% b) 25% c) 10% d) 15%



8. A birdwatcher sights an eagle at the top of a 20 m tree. The birdwatcher is lying on the ground 50 m from the tree. At what angle must he incline his camera to take a photograph of the eagle? Give the answer to the nearest degree.

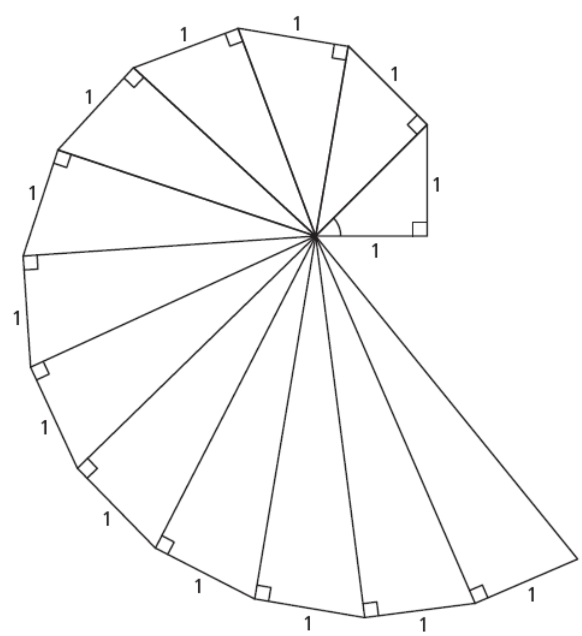
9. The Pioneer ski lift at Golden, B.C., is 1366 m long. It rises 522 m vertically. What is the angle of inclination of the ski lift? Give the answer to the nearest degree.

10. From a rectangular board, a carpenter cuts a stringer to support some stairs. Each stair rises 7.5 in. and has a tread of 11.0 in. To the nearest degree, at what angle should the carpenter cut the board?



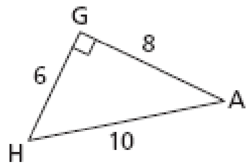
11. For the tangent of an acute angle in a right triangle: a)What is the least possible value? b)What is the greatest possible value? Justify your answers.

12. A Pythagorean spiral is constructed by drawing right triangles on the hypotenuse of other right triangles. Start with a right triangle in which each leg is 1 unit long. Use the hypotenuse of that triangle as one leg of a new triangle and draw the other leg 1 unit long. Continue the process. A spiral is formed.
   a) Determine the tangent of the angle at the centre of the spiral in each of the first 5 triangles.
   b) Use the pattern in part a to predict the tangent of the angle at the centre of the spiral for the 100th triangle.

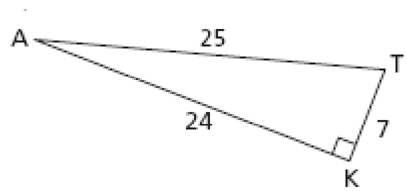


13. a) In each triangle below:
Name the side opposite ∠A.
Name the side adjacent to ∠A.
Name the hypotenuse.

i)



ii)



b) For each triangle in part a, determine sin A and cos A to the nearest hundredth

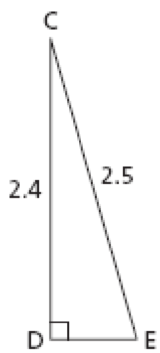
14. For each ratio below, sketch two different right triangles and label their sides.

a) $\sin B = \frac{3}{5}$ b) $\cos B = \frac{5}{8}$

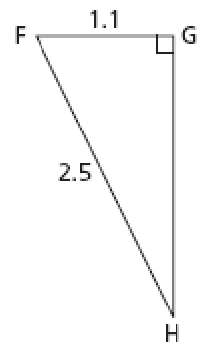
c) $\sin B = \frac{1}{4}$ d) $\cos B = \frac{4}{9}$

15. Use the sine or cosine ratio to determine the measure of each acute angle to the nearest tenth of a degree.

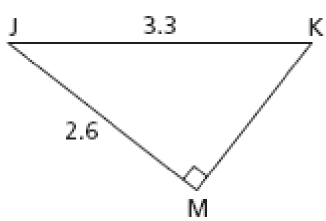
a)



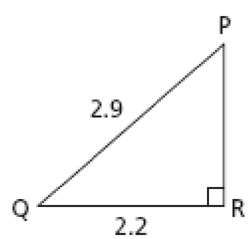
b)



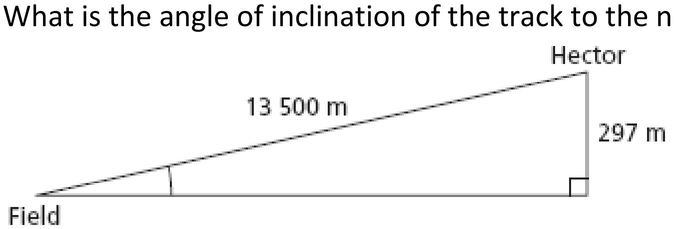
c)



d)



16. Suppose the railroad track through the spiral tunnels from Field to Hector were straightened out. It would look like the diagram below. The diagram is *not* drawn to scale. What is the angle of inclination of the track to the nearest tenth of a degree?



17. A ladder is 6.5 m long. It leans against a wall. The base of the ladder is 1.2 m from the wall. What is the angle of inclination of the ladder to the nearest tenth of a degree?

18. A rope that supports a tent is 2.4 m long. The rope is attached to the tent at a point that is 2.1 m above the ground. What is the angle of inclination of the rope to the nearest degree?

19. A rectangle is 4.8 cm long and each diagonal is 5.6 cm long. What is the measure of the angle between a diagonal and the longest side of the rectangle? Give the answer to the nearest degree.

20. a) Calculate:

i) sin 10°	ii) sin 20°	iii) sin 40°
iv) sin 50°	v) sin 60°	vi) sin 80°

b) Why does the sine of an angle increase as the angle increases?

21. Sketch a right isosceles triangle. Explain why the cosine of each acute angle is equal to the sine of the angle.