Math 10

Lesson 2-3 Factoring trinomials

I. Lesson Objectives:

a) To see the patterns in multiplying binomials that can be used to factor trinomials into binomials.
b) To factor trinomials of the form $ax^2 + bx + c$.

II. Binomial multiplication – hunting for patterns

In the previous lesson we saw how the distributive property could be used to multiply binomials together. In this lesson we are interested in doing the reverse – **we want to factor trinomials into binomials**. Perhaps if we studied how binomials multiply together we can find some patterns that may help us to reverse the process when we factor a trinomial. With this in mind, let's do a few more binomial multiplications to see if any pattern(s) become evident. Consider the following five binomial multiplications:

$(x + 3)(x + 2)$	$(x - 3)(x + 2)$	$(2x - 3)(x + 2)$	$(2x - 3)(5x - 2)$	$(x + 3)(x - 3)$
$= x(x + 2) + 3(x + 2)$	$= x(x + 2) - 3(x + 2)$	$= 2x(x + 2) - 3(x + 2)$	$= 2x(5x - 2) - 3(5x - 2)$	$= x(x - 3) + 3(x - 3)$
$= x^2 + 2x + 3x + 6$	$= x^2 + 2x - 3x - 6$	$= 2x^2 + 4x - 3x - 6$	$= 10x^2 - 4x - 15x + 6$	$= x^2 - 3x + 3x - 9$
$= x^2 + 5x + 6$	$= x^2 - x - 6$	$= 2x^2 + x - 6$	$= 2x^2 - 19x + 6$	$= x^2 - 9$

The first pattern is that the **first two terms of the binomials multiply together to form the first term of each trinomial**.

$(\mathbf{x} + 3)(\mathbf{x} + 2)$	$(\mathbf{x} - 3)(\mathbf{x} + 2)$	$(\mathbf{2x} - 3)(\mathbf{x} + 2)$	$(\mathbf{2x} - 3)(\mathbf{5x} - 2)$	$(\mathbf{x} + 3)(\mathbf{x} - 3)$
$= x(x + 2) + 3(x + 2)$	$= x(x + 2) - 3(x + 2)$	$= 2x(x + 2) - 3(x + 2)$	$= 2x(5x - 2) - 3(5x - 2)$	$= x(x - 3) + 3(x - 3)$
$= x^2 + 2x + 3x + 6$	$= x^2 + 2x - 3x - 6$	$= 2x^2 + 4x - 3x - 6$	$= 10x^2 - 4x - 15x + 6$	$= x^2 - 3x + 3x - 9$
$= \mathbf{x^2} + 5x + 6$	$= \mathbf{x^2} - x - 6$	$= \mathbf{2x^2} + x - 6$	$= \mathbf{10x^2} - 19x + 6$	$= \mathbf{x^2} - 9$

The second pattern is that the **last two terms of the binomials multiply together to form the last term of each trinomial**.

$(x \mathbf{+ 3})(x \mathbf{+ 2})$	$(x \mathbf{- 3})(x \mathbf{+ 2})$	$(2x \mathbf{- 3})(x \mathbf{+ 2})$	$(2x \mathbf{- 3})(5x \mathbf{- 2})$	$(x \mathbf{+ 3})(x \mathbf{- 3})$
$= x(x + 2) + 3(x + 2)$	$= x(x + 2) - 3(x + 2)$	$= 2x(x + 2) - 3(x + 2)$	$= 2x(5x - 2) - 3(5x - 2)$	$= x(x - 3) + 3(x - 3)$
$= x^2 + 2x + 3x + 6$	$= x^2 + 2x - 3x - 6$	$= 2x^2 + 4x - 3x - 6$	$= 10x^2 - 4x - 15x + 6$	$= x^2 - 3x + 3x - 9$
$= x^2 + 5x \mathbf{+ 6}$	$= x^2 - x \mathbf{- 6}$	$= 2x^2 + x \mathbf{- 6}$	$= 2x^2 - 19x \mathbf{+ 6}$	$= x^2 \mathbf{- 9}$

The third pattern is that the **middle term of the trinomial is formed when we add the x-terms together**.

$(x + 3)(x + 2)$	$(x - 3)(x + 2)$	$(2x - 3)(x + 2)$	$(2x - 3)(5x - 2)$	$(x + 3)(x - 3)$
$= x(x + 2) + 3(x + 2)$	$= x(x + 2) - 3(x + 2)$	$= 2x(x + 2) - 3(x + 2)$	$= 2x(5x - 2) - 3(5x - 2)$	$= x(x - 3) + 3(x - 3)$
$= x^2 \mathbf{+ 2x + 3x} + 6$	$= x^2 \mathbf{+ 2x - 3x} - 6$	$= 2x^2 \mathbf{+ 4x - 3x} - 6$	$= 10x^2 \mathbf{- 4x - 15x} + 6$	$= x^2 \mathbf{- 3x + 3x} - 9$
$= x^2 \mathbf{+ 5x} + 6$	$= x^2 \mathbf{- x} - 6$	$= 2x^2 \mathbf{+ x} - 6$	$= 2x^2 \mathbf{- 19x} + 6$	$= x^2 + \mathbf{0x} - 9$

The fourth pattern is a little harder to see, but it leads directly to something we can use. The pattern can be seen in the third line of each binomial multiplication. When we **multiply the coefficients of the middle terms** and we **multiply the end term coefficients**, we get the **same number**!!

$(x + 3)(x + 2)$	$(x - 3)(x + 2)$	$(2x - 3)(x + 2)$	$(2x - 3)(5x - 2)$	$(x + 3)(x - 3)$
$= x(x + 2) + 3(x + 2)$	$= x(x + 2) - 3(x + 2)$	$= 2x(x + 2) - 3(x + 2)$	$= 2x(5x - 2) - 3(5x - 2)$	$= x(x - 3) + 3(x - 3)$
$= \underline{1}x^2 \mathbf{+ 2}x \mathbf{+ 3}x \underline{+ 6}$	$= \underline{1}x^2 \mathbf{+ 2}x \mathbf{- 3}x \underline{- 6}$	$= \underline{2}x^2 \mathbf{+ 4}x \mathbf{- 3}x \underline{- 6}$	$= \underline{10}x^2 \mathbf{- 4}x \mathbf{- 15}x + \underline{6}$	$= \underline{1}x^2 \mathbf{- 3}x \mathbf{+ 3}x \underline{- 9}$
$= x^2 + 5x + 6$	$= x^2 - x - 6$	$= 2x^2 + x - 6$	$= 2x^2 - 19x + 6$	$= x^2 - 9$
$\mathbf{2} \cdot \mathbf{3} = 6$	$\mathbf{2} \cdot \mathbf{-3} = -6$	$\mathbf{4} \cdot \mathbf{-3} = -12$	$\mathbf{-4} \cdot \mathbf{-15} = 60$	$\mathbf{-3} \cdot \mathbf{3} = -9$
$\underline{1} \cdot \underline{6} = 6$	$\underline{1} \cdot \underline{-6} = -6$	$\underline{2} \cdot \underline{-6} = -12$	$\underline{10} \cdot \underline{6} = 60$	$\underline{1} \cdot \underline{-9} = -9$

Ah ha!! or, as Archimedes would have said, "Eureka!!". When we multiply the end terms of a trinomial together and then write its factors, two of the factors add to form the middle term coefficient. Thus we have a pattern that we can use to factor a trinomial:

> To factor any trinomial of the form $ax^2 + bx + c$, decompose bx into two terms whose coefficients have a <u>sum</u> of b and a <u>product</u> equal to $a \cdot c$.

(That is one tough sentence to interpret!!) The idea becomes simpler when we break it down into a few steps and then show some examples.

The basic steps for factoring trinomials with the form $ax^2 + bx + c$, are:

1) Multiply $a \cdot c$ to produce <u>the number</u>.
2) List the factors of <u>the number</u>.
3) Find two factors of <u>the number</u> that add up to b.
4) Decompose bx into the two factors.
5) Factor the polynomial by grouping.

If this cannot be found, the trinomial cannot be factored by this method.

The word **composition** means "to bring together." Therefore, to **decompose** something is to split it apart.

III. Factoring trinomials – examples

The basic steps are reproduced below so you do not have to flip pages back and forth.

The basic steps for factoring trinomials with the form $ax^2 + bx + c$, are:

1) Multiply $a \cdot c$ to produce the number.
2) List the factors of the number.
3) Find two factors of the number that add up to b.
4) Decompose bx into the two factors.
5) Factor the polynomial by grouping.

If this cannot be found, the trinomial cannot be factored by this method.

The word **composition** means "to bring together." Therefore, to **decompose** something is to split it apart.

Let's look at several examples to get the idea.

Example 1 Factor $x^2 + 5x + 4$ using (a) decomposition and (b) algebra tiles

a)

$a = 1$ and $c = 4 \therefore a \cdot c = 4$

Find two integers with a product of 4 and a sum of 5

Factors of 4	Sum
1 x 4	5
2 x 2	4

The product is positive and the sum is positive. What signs do the integers need to have?

$x^2 + 5x + 4$

$= x^2 + 4x + x + 4$

$= (x^2 + 4x) + (x + 4)$

$= x(x + 4) + 1(x + 4)$

$= (x + 1)(x + 4)$

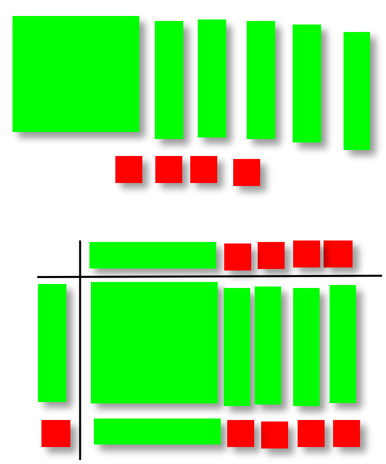
Decompose the middle term into its factors (1 & 4).

Factor by grouping the first two terms and the last two terms.

b)

Arrange one x^2-tile, five x-tiles and four 1-tiles into a rectangle. Then place tiles around the rectangle to show its dimensions.
The dimensions of the rectangle are $x + 4$ and $x + 1$.

$\therefore\ x^2 + 5x + 4 = (x + 4)(x + 1)$



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Example 2 Factor $3x^2 + 8x + 4$ using (a) decomposition and (b) algebra tiles

When there are coefficients other than 1, it is wise to check if there is a common factor that can be removed. In this case there is no common factor for 3, 8 and 4.

a)

$a \cdot c = 3 \cdot 4 = 12$

The product is positive and the sum is positive.
∴ both factors will be positive

Find two integers with a product of 4 and a sum of 5

Factors of 4	Sum
1 x 12	13
2 x 6	8
3 x 4	12

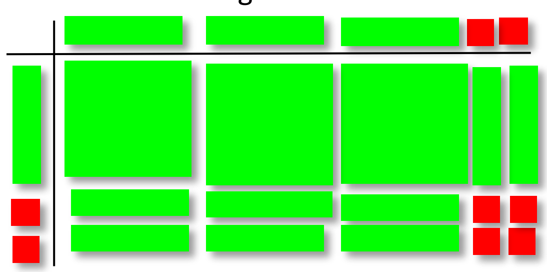
$3x^2 + 8x + 4$
$= 3x^2 + 2x + 6x + 4$
$=(3x^2 + 2x) + (6x + 4)$
$= x(3x + 2) + 2(3x + 2)$
$= (x + 2)(3x + 2)$

Decompose the middle term into its factors (2 & 6).

Factor by grouping the first two terms and the last two terms.

b)

Arrange three x^2-tiles, eight x-tiles and four 1-tiles into a rectangle. Then place tiles around the rectangle to show its dimensions.



The dimensions of the rectangle are 3x + 2 and x + 2.

Check
$(3x + 2)(x + 2)$
$= 3x(x + 2) + 2(x + 2)$
$= 3x^2 + 6x +2x + 4$
$= 3x^2 + 8x + 4$

Example 3 Factor $24x^2 – 30x – 9$

A check for the greatest common factor of 24, 30 and 9 reveals 3 as the GCF.

$24x^2 – 30x – 9$
$= 3(8x^2 – 10x – 3)$
$= 3(8x^2 – 12x + 2x – 3)$
$= 3[(8x^2 – 12x) + (2x – 3)]$
$= 3[4x(2x – 3) + 1(2x – 3)]$
$= \mathbf{3(4x + 1)(2x – 3)}$

$a \cdot c = 8 \cdot –3 = –24$

The product is negative and the sum is negative. ∴ one factor will be negative

Find two integers with a product of 24 and a sum of –10

Factors of –24	Sum
–1 x 24	23
–2 x 12	10
2 x –12	–10

We note that the sum is correct accept for sign.

Reverse the signs on the factors.

Question 1

If possible, factor each trinomial.

a) $x^2 + 5x + 6$ b) $x^2 - 29x + 28$ c) $x^2 - 3xy - 18y^2$

Question 2

If possible, factor each trinomial

a) $2x^2 + 7x - 4$ b) $-3s^2 - 51s - 30$ c) $3x^2 + x - 4$

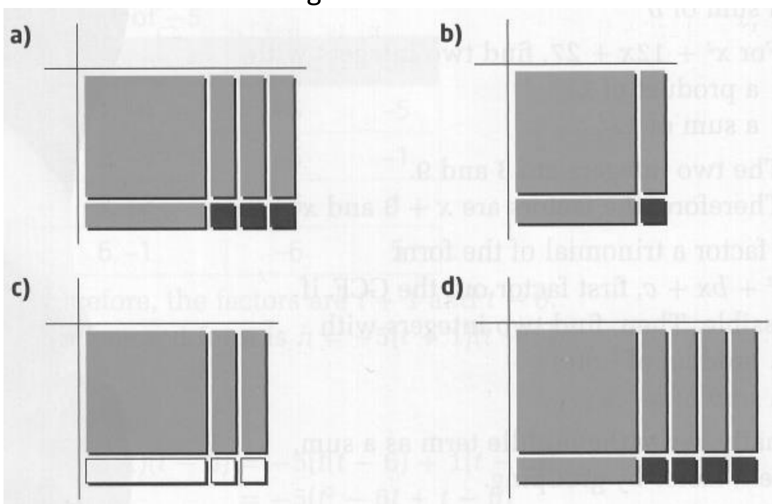
Question 3

If possible, factor each trinomial

a) $x^2 + 7x + 10$ b) $6x^2 - 5xy + y^2$ c) $2y^2 + 7xy + 3x^2$

IV. Assignment

1. Write the trinomial represented by each rectangle of algebra tiles. Then, determine the dimensions of each rectangle.



2. Factor each trinomial.
   a) $2x^2 + 5x + 3$
   b) $3x^2 + 7x + 4$
   c) $3x^2 + 7x - 6$
   d) $6x^2 + 11x + 4$

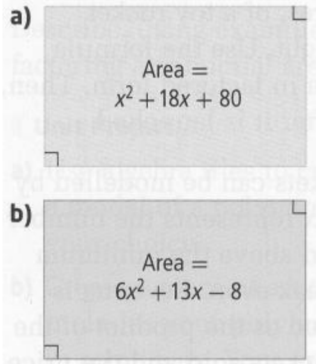
3. Factor, if possible.
   a) $x^2 + 7x + 10$ b) $j^2 + 12j + 27$
   c) $k^2 + 5k + 4$ d) $p^2 + 9p + 12$
   e) $d^2 + 10d + 24$ f) $c^2 + 4cd + 21d^2$

4. Factor each trinomial.
   a) $m^2 - 7m + 10$ b) $s^2 + 3s - 10$
   c) $f^2 - 7f + 6$ d) $g^2 - 5g - 14$
   e) $b^2 - 3b - 4$ f) $2r^2 - 14rs + 24s^2$

5. Factor, if possible.
   a) $2x^2 + 7x + 5$ b) $6y^2 + 19y + 8$
   c) $3m^2 + 10m + 8$ d) $10w^2 + 15w + 3$
   e) $12q^2 + 17q + 6$ f) $3x^2 + 7xy + 2y^2$

6. Factor, if possible.
   a) $4x^2 - 11x + 6$ b) $w^2 + 11w + 25$
   c) $x^2 - 5x + 6$ d) $2m^2 + 3m - 9$
   e) $6x^2 - 3xy - 3y^2$ f) $12y^2 + y - 1$
   g) $6c^2 + 7cd - 10d^2$ h) $4k^2 + 15k + 9$
   i) $a^2 + 11ab + 24b^2$ j) $6m^2 + 13mn + 2n^2$

7. Identify binomials that represent the length and width of each rectangle. Then, calculate the dimensions of the rectangle if x = 15 cm.



8. You can estimate the height, h, in metres, of a toy rocket at any time, t, in seconds, during its flight. Use the formula $h = -5t^2 + 23t + 10$. Write the formula in factored form. Then, calculate the height of the rocket 3 s after it is launched.

9. You have been asked to factor the expression $30x^2 - 39xy - 9y^2$. What are the factors?

10. A rescue worker launches a signal flare into the air from the side of a mountain. The height of the flare can be represented by the formula $h = -16t^2 + 144t + 160$. In the formula, *h* is the height, in feet, above ground, and *t* is the time, in seconds.
    a) What is the factored form of the formula?
    b) What is the height of the flare after 5.6 s?